Metric Entropy of Integration Operators and Small Ball Probabilities for the Brownian Sheet

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Let T_d: $L_2([0, 1]^d) \to C([0, 1]^d)$ be the d-dimensional integration operator. We show that its Kolmogorov and entropy numbers decrease with order at least $k^{-1}(\log k)^{d-1/2}$. From this we derive that the small ball probabilities of the Brownian sheet on $[0, 1]^d$ under the $C([0, 1]^d)$-norm can be estimated from below by $\exp(-C\varepsilon^{-2} |\log \varepsilon|^{2d-1})$, which improves the best known lower bounds considerably. We also get similar results with respect to certain Orlicz norms. © 1999 Academic Press

1. INTRODUCTION

Let us consider the *d*-dimensional integration operator T_d: $L_2([0, 1]^d) \to C([0, 1]^d)$ defined as

$$T_d f(x_1, \ldots, x_d) := \int_0^{x_1} \cdots \int_0^{x_d} f(y_1, \ldots, y_d) \, dy_d \cdots dy_1, \qquad f \in L_2([0, 1]^d).$$

One can view T_d as the *d*-fold tensor product $T \otimes \cdots \otimes T$ of the usual one-dimensional integration operator $T = T_1$. Let $C, C_1, \ldots$ denote constants

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(different at each occurrence) which either are universal or depend on the dimension d only. Our main result concerning T_d is the following estimate for its Kolmogorov and entropy numbers.

THEOREM 1.   *There exist constants C_1 and C_2 such that*

$$d_k(T_d) \leqslant \frac{C_1}{k}(1 + \log k)^{d-1/2} \qquad and \qquad e_k(T_d) \leqslant \frac{C_2}{k}(1 + \log k)^{d-1/2}$$

hold for all $k \geqslant 1$.

It is known that the Kolmogorov and entropy numbers above can be estimated from below by $Ck^{-1}(\log k)^{d-1}$. This fact can be proved, e.g., by considering the operator T_d mapping into $L_2([0, 1]^d)$ instead of $C([0, 1]^d)$ and combining the results of [13] and [8].

In the language of the theory of function spaces, $d_k(T_d)$ and $e_k(T_d)$ are essentially equivalent to Kolmogorov and entropy numbers for Sobolev classes of functions with L_2-bounded mixed derivative. In this setting, the question was considered by Temlyakov (see, e.g., [25–27]). His result (see, e.g., Theorem 3.3 from [27]) contains the statement of Theorem 1 in the case $d = 2$, but for $d \geqslant 3$ our estimate is better than its counterpart. We refer to [11] for entropy bounds of various more classical Sobolev classes.

A motivation for our research came recently from the theory of probability. In their remarkable paper [13], Kuelbs and Li showed that for each Gaussian measure the measures of small balls are closely connected with the metric entropy of a linear operator. It turns out that the operator T_d is related in this way to the distribution of the Brownian sheet, a very important Gaussian random field (for the definition see Section 3). Therefore, our theorem yields substantial progress in the difficult question about the small ball probabilities of the Brownian sheet (see Theorem 6 below). For a refined exposition and further development of the ideas of Kuelbs and Li, we refer the interested reader to [14, Section 7].

The paper is organized as follows. Section 2 is devoted to the operator T_d and contains the estimates of its Kolmogorov and entropy numbers. In this section we also consider the operator T_d under certain Orlicz norms. The lower bounds for the small ball probabilities of the Brownian sheet are proved in Section 3.

2. ENTROPY ESTIMATES FOR THE INTEGRATION OPERATOR

2.1. *Basic Notions*

First, we recall some definitions that are needed throughout this section. Let $S: E \to F$ be a compact operator between Banach spaces. Denote by B_E

and B_F the unit balls of E and F, respectively. The covering numbers of a pre-compact set $C \subset F$ are defined by

$$N(\varepsilon, C) := \min \left\{ k \geqslant 1 : \exists x_1, \dots, x_k \in C \text{ such that } C \subset \bigcup_{j=1}^{k} (x_j + \varepsilon B_F) \right\}$$

and the metric entropy of C is $H(\varepsilon, C) := \log N(\varepsilon, C)$. The (dyadic) entropy numbers can be regarded as inverse function of H. They are defined by

$$e_k(C) := \inf\{\varepsilon > 0 : N(\varepsilon, C) \leqslant 2^{k-1}\}$$

and we write $e_k(S)$ instead of $e_k(SB_E)$. For any closed subspace $\tilde{F} \subset F$ we denote by $Q_{\tilde{F}}$ the quotient mapping from F onto $F/\tilde{F}$. Then the Kolmogorov numbers of S are defined as

$$d_k(S) := \inf\{\|Q_{\tilde{F}}S\| : \tilde{F} \subset F \text{ with dim } \tilde{F} < k\}$$
$$= \inf\{\varepsilon > 0 : \exists \tilde{F} \subset F \text{ with dim } \tilde{F} < k \text{ and } SB_E \subset \tilde{F} + \varepsilon B_F\}.$$

Finally, we have to introduce the ℓ-norm of an operator S mapping a Hilbert space H into a Banach space, which is defined as

$$\ell(S) := \sup \left\{ \left(\mathbb{E} \left\| \sum_{j=1}^{n} \xi_j Sf_j \right\|^2 \right)^{1/2} : n \in \mathbb{N}; f_1, \dots, f_n \in H \text{ orthonormal} \right\},$$

where $\xi_1, \xi_2, \dots$ are independent $\mathcal{N}(0, 1)$-distributed random variables. The symbol $\mathbb{E}$ stands for the mathematical expectation and in the case above it is nothing but the integral $(\int_{\mathbb{R}^n} \|\sum_1^n x_j Sf_j\|^2 \mu_n(dx))^{1/2}$, where μ_n is the standard Gaussian measure on $\mathbb{R}^n$.

Next, we recall some properties of the approximation quantities defined above. Let us consider an operator S mapping a Hilbert space into a Banach space. We need the following two estimates. A result of Pajor and Tomczak–Jaegermann [19] provides a relation between Kolmogorov numbers and ℓ-norm, namely

$$\sup_{k \geqslant 1} k^{1/2} d_k(S) \leqslant C\ell(S) \tag{1}$$

(cf. [21, Theorem 5.8]). Moreover, we can easily deduce the same inequality for entropy numbers using Carl's inequality [5]

$$\sup_{k \geqslant 1} k^{1/2} e_k(S) \leqslant C \sup_{k \geqslant 1} k^{1/2} d_k(S). \tag{2}$$

In what follows, we split the operator T_d into finite dimensional parts. Using probabilistic arguments, we give upper bounds for the ℓ-norm of each part and estimate Kolmogorov and entropy numbers via (1) and (2).

Remark. As an alternative approach to the entropy estimate one could use Sudakov's result (cf. [21, Theorem 5.5])

$$\sup_{k \geqslant 1} k^{1/2} e_k(S^*) \leqslant C\ell(S),$$

where S^* denotes the dual operator of S, and combine it with the duality bounds of Tomczak-Jaegermann [29], which provide the links between $e_k(S^*)$ and $e_k(S)$. For probabilistic applications, which involve only the entropy numbers, this would be sufficient.

2.2 *Multidimensional Haar Basis*

The Haar basis in $L_2[0, 1]$ consists of the function $h_{-1, 0} = \mathbf{1}_{[0, 1]}$ and, for $m \geqslant 0$, of the functions

$$h_{m, i}(x) := 2^{m/2} h(2^m(x - i2^{-m})),$$

with $i = 0, ..., 2^m - 1$, where $h := \mathbf{1}_{[0, 1/2)} - \mathbf{1}_{[1/2, 1)}$. Denote by $J_m := \{0, ..., 2^m - 1\}$ the index set related to m, for $m \geqslant 0$, and $J_{-1} := \{0\}$. Defining $u_{m, i} := Th_{m, i}$, we observe

$$u_{m, i}(x) = 2^{-m/2} u(2^m(x - i2^{-m}))$$

for $m \geqslant 0$, where $u(x) := x\mathbf{1}_{[0, 1/2)} + (1 - x) \mathbf{1}_{[1/2, 1)}$. Consequently, for each fixed $m \geqslant 0$ the sets $\{x \in [0, 1] : u_{m, i}(x) \neq 0\}$, for $i \in J_m$, are disjoint open intervals of length 2^{-m} and for the supremum of $u_{m, i}$ we have $\|u_{m, i}\|_{C([0, 1])} = 2^{-1 - m/2}$.

In what follows, let m denote a multi-index $(m_1, ..., m_d) \in \{-1, 0, 1, ...\}^d$. We define

$$|m| := \sum_{j=1}^{d} \max(m_j, 0)$$

and introduce the sets $M_n := \{m : |m| = n\}$, $n = 0, 1, ...$. One can verify that the cardinality of M_n is of order n^{d-1}; i.e., $\#(M_n) \asymp n^{d-1}$ for $n \to \infty$. Let $J_m := J_{m_1} \times \cdots \times J_{m_d}$ be the set of all indices which correspond to the index m. Defining

$$h_{m, i} := h_{m_1, i_1} \otimes \cdots \otimes h_{m_d, i_d} \qquad \text{and} \qquad u_{m, i} := u_{m_1, i_1} \otimes \cdots \otimes u_{m_d, i_d},$$

for all $m \in \{-1, 0, 1, ...\}^d$ and $i \in J_m$, we introduce the subspaces

$$H_n := \text{span}\{h_{m, i} : m \in M_n; i \in J_m\} \subset L_2([0, 1]^d), \qquad \text{for} \quad n = 0, 1, ...$$

Clearly $L_2([0, 1]^d) = \bigoplus_{n=0}^{\infty} H_n$. We denote by P_n the orthogonal projector from $L_2([0, 1]^d)$ onto H_n. Then the operator T_d admits the representation $T_d = \sum_{n=0}^{\infty} R_n$ with $R_n := T_d P_n$.

2.3. Key Estimate

The following proposition contains one of our main tools.

PROPOSITION 2. *Let* $R_n : L_2([0, 1]^d) \rightarrow C([0, 1]^d)$ *be defined as above. Then we have the estimate* $\ell(R_n) \leqslant C n^{d/2} 2^{-n/2}$ *with a constant C depending only on d.*

Proof. Recall that by definition of ℓ,

$$\ell(R_n) = \left(\mathbb{E} \sup_{t \in [0, 1]^d} \left| \sum_{m \in M_n} \sum_{i \in J_m} \xi_{m, i} u_{m, i}(t) \right|^2 \right)^{1/2},$$

where the r.v.'s $(\xi_{m, i})$ are independent and $\mathcal{N}(0, 1)$-distributed. By equivalence of Gaussian norms (cf. [21, Corollary 4.9]) one can omit square and square root at the cost of a universal constant:

$$\ell(R_n) \leqslant C \, \mathbb{E} \sup_{t \in [0, 1]^d} \left| \sum_{m \in M_n} \sum_{i \in J_m} \xi_{m, i} u_{m, i}(t) \right|.$$

In a first step we apply a time discretization procedure to the stochastic process

$$B_n(t) := \sum_{m \in M_n} \sum_{i \in J_m} \xi_{m, i} u_{m, i}(t), \qquad t \in [0, 1]^d.$$

For this purpose, consider the grid $G_n := \{(2i + 1) \, 2^{-2n-1} : i = 0, \dots, 2^{2n}\}^d$. Each $x \in G_n$ is the center of a cube $\kappa_x := \prod_{j=1}^{d} [i_j 2^{-2n}, (i_j + 1) \, 2^{-2n}]$ where the i_j's are suitably chosen. Obviously, these cubes cover $[0, 1]^d$; therefore we have

$$\mathbb{E} \sup_{t \in [0, 1]^d} |B_n(t)| \leqslant \mathbb{E} \sup_{x \in G_n} \sup_{t \in \kappa_x} |B_n(t) - B_n(x)| + \mathbb{E} \sup_{x \in G_n} |B_n(x)|.$$

For the first summand we use the trivial estimate

$$|B_n(t) - B_n(x)| \leqslant \max_{m, i} |\xi_{m, i}| \sum_{m \in M_n} \sum_{i \in J_m} |u_{m, i}(t) - u_{m, i}(x)|.$$

Since for fixed $m \in M_n$ the functions $u_{m, i}$, $i \in J_m$, have pairwise essentially disjoint supports, and every κ_x is contained in one of these supports, the

sum over i reduces to a single summand. On every κ_x, moreover, all functions $u_{m,i}$ are differentiable and their gradients satisfy

$$\sup_{t \in \kappa_x} |\nabla u_{m,i}(t)|_2 \leqslant C 2^{n/2}$$

(here $|\cdot|_2$ denotes the Euclidean norm on $\mathbb{R}^d$). Now the mean value theorem yields

$$\sup_{t \in \kappa_x} |u_{m,i}(x) - u_{m,i}(t)| \leqslant C 2^{n/2} \sup_{t \in \kappa_x} |x - t|_2 \leqslant C 2^{-3n/2}.$$

Using finally $\mathbb{E} \max_{m,i} |\xi_{m,i}| \leqslant C \sqrt{1 + \log N}$ with $N := \#(\bigcup_{m \in M_n} J_m)$ (see, e.g., [21, Lemma 4.14]) we obtain

$$\mathbb{E} \sup_{x \in G_n} \sup_{t \in \kappa_x} |B_n(t) - B_n(x)| \leqslant C \#(M_n) \, 2^{-3n/2} \sqrt{1 + \log N} \leqslant C n^{d-1/2} 2^{-3n/2}.$$

The second summand can be treated similarly. Again we have by Lemma 4.14 of [21]

$$\begin{aligned}
\mathbb{E} \sup_{x \in G_n} |B_n(x)| &\leqslant C \sqrt{1 + \log \#(G_n)} \max_{x \in G_n} (\mathbb{E} |B_n(x)|^2)^{1/2} \\
&\leqslant C \sqrt{1 + \log \#(G_n)} \sqrt{\#(M_n)} \, 2^{-n/2} \\
&\leqslant C n^{d/2} 2^{-n/2}.
\end{aligned}$$

Here we used the estimate $\|u_{m,i}\|_{C([0,1]^d)} \leqslant 2^{-n/2}$ for all $m \in M_n$, $i \in J_m$. The proof is finished. ∎

Remark. Although the estimate $\ell(R_n) \leqslant C n^{d/2} 2^{-n/2}$ seems to be quite rough, it is asymptotically sharp. Since we do not need this fact here, we omit the proof (for a proof see [9]).

Proof of Theorem 1. Let S be an operator between Banach spaces. Suppose $\sup_k k^{1/p} d_k(S) < \infty$, $p \in (0, \infty)$, then its quasi-norm $\mathbf{L}_{p,\infty}^{(d)}(S)$ is defined by

$$\mathbf{L}_{p,\infty}^{(d)}(S) := \sup_{k \geqslant 1} k^{1/p} d_k(S).$$

Similarly, one defines $\mathbf{L}_{p,\infty}^{(e)}(S)$ replacing Kolmogorov numbers by entropy numbers (cf. [20] or [7]). Recall that rank $R_n =: N \asymp n^{d-1} 2^n$. Then, for an arbitrary fixed $p \in (0, 2)$, Proposition 2 and (1) yield

$$\mathbf{L}_{p,\infty}^{(d)}(R_n) = \sup_{1 \leqslant k \leqslant N} k^{1/p} d_k(R_n) \leqslant C n^{(d-1)/p + 1/2} 2^{-n(1-1/p)}.$$

For any given integer $K \in \mathbb{N}$ we split the operator T_d as the sum of

$$\underline{T_d} := \sum_{n=0}^{K} R_n \qquad \text{and} \qquad \overline{T_d} := \sum_{n=K+1}^{\infty} R_n.$$

Let us fix a $p \in (1, 2)$. The quasi-norm $\mathbf{L}_{p,\infty}^{(d)}$ is equivalent to an r-norm, for some $r \in (0, 1)$; hence

$$\begin{aligned}
\mathbf{L}_{p,\infty}^{(d)}(\overline{T_d}) &\leqslant C \left(\sum_{n=K+1}^{\infty} (\mathbf{L}_{p,\infty}^{(d)}(R_n))^r \right)^{1/r} \\
&\leqslant C \left(\sum_{n=K+1}^{\infty} (n^{(d-1)/p+1/2} 2^{-n(1-1/p)})^r \right)^{1/r} \\
&\leqslant C K^{(d-1)/p+1/2} 2^{-K(1-1/p)},
\end{aligned}$$

which implies

$$d_{K^{d-1} 2^K}(\overline{T_d}) \leqslant C K^{1/2} 2^{-K}.$$

Using the estimate $\operatorname{rank} \underline{T_d} \leqslant C \sum_{n=1}^{K} n^{d-1} 2^n \leqslant C K^{d-1} 2^K$, we have $d_{CK^{d-1} 2^K}(\underline{T_d}) = 0$, and hence the additivity of Kolmogorov numbers gives

$$d_{CK^{d-1} 2^K}(T_d) \leqslant C K^{1/2} 2^{-K}$$

for all $K \in \mathbb{N}$. Then the monotonicity of Kolmogorov numbers implies

$$d_k(T_d) \leqslant \frac{C}{k} (1 + \log k)^{d-1/2}$$

for all $k \in \mathbb{N}$. Since $\mathbf{L}_{p,\infty}^{(e)}(R_n) \leqslant c_p \mathbf{L}_{p,\infty}^{(d)}(R_n)$ by Carl's inequality (2), the same arguments (with some modifications for T_d) prove the statement on entropy numbers. This method follows closely ideas which can already be found in [6]. ∎

One can easily reformulate the result above in terms of metric entropy.

COROLLARY 3. *Let $B_{L_2([0,1]^d)}$ be the closed unit ball of $L_2([0,1]^d)$. There exists a constant C such that*

$$H(\varepsilon, T_d(B_{L_2([0,1]^d)})) \leqslant \frac{C}{\varepsilon} |\log \varepsilon|^{d-1/2}$$

for sufficiently small $\varepsilon > 0$.

2.4. *Orlicz Norms*

For $2 \leqslant p < \infty$ consider the Orlicz function $\psi_p(t) := \exp(t^p) - 1$, $t \geqslant 0$. Then the Orlicz space $L_{\psi_p}([0, 1]^d)$ consists of all measurable functions f on $[0, 1]^d$ with finite Orlicz norm

$$\|f\|_{\psi_p} := \inf \left\{ c > 0 : \int_{[0, 1]^d} \psi_p(|f(x)|/c) \, dx \leqslant 1 \right\}.$$

Our strategy will be the same as for the sup-norm and we also keep the previous notations. Our first result is in complete analogy with Proposition 2.

PROPOSITION 4. *For* $R_n : L_2([0, 1]^d) \to L_{\psi_p}([0, 1]^d)$ *we have* $\ell(R_n) \leqslant C n^{d/2 - 1/p} 2^{-n/2}$, *where the constant C depends only on $d \geqslant 2$ and $p \in [2, \infty)$.*

Proof. In a first step we treat the case $p = 2$. For every fixed $x \in [0, 1]^d$ the random variable

$$B_n(x, \omega) := \sum_{m \in M_n} \sum_{i \in J_m} \xi_{m, i}(\omega) \, u_{m, i}(x)$$

has the same distribution as $\sigma(x) \xi$, where $\sigma(x)^2 := \sum_{m \in M_n} \sum_{i \in J_m} u_{m, i}(x)^2$ and $\xi \sim \mathcal{N}(0, 1)$. Because of $\|u_{m, i}\|_{C([0, 1]^d)} \leqslant 2^{-n/2}$ and since for fixed $m \in M_n$ the functions $u_{m, i}$ are disjointly supported, we can estimate

$$\sigma(x) \leqslant \sigma := \sqrt{\#(M_n)} \, 2^{-n/2} \asymp n^{(d-1)/2} 2^{-n/2}.$$

Therefore, for every $\lambda \geqslant 1$, we have

$$\begin{aligned}
\mathbb{E} \, \psi_2(|B_n(x)|/2\sigma\lambda) &= \mathbb{E} \exp(\xi^2 \sigma(x)^2 / 4\sigma^2\lambda^2) - 1 \\
&\leqslant \mathbb{E} \exp(\xi^2/4\lambda^2) - 1 \\
&= \left(1 - \frac{1}{2\lambda^2}\right)^{-1/2} - 1 \leqslant \frac{1}{\lambda^2}.
\end{aligned}$$

Integrating x over $[0, 1]^d$ and using Fubini's theorem give

$$\mathbb{E}_\omega \int_{[0, 1]^d} \psi_2(|B_n(x, \omega)|/2\sigma\lambda) \, dx \leqslant \frac{1}{\lambda^2}.$$

Whenever $\|B_n(\cdot, \omega)\|_{\psi_2} > 2\sigma\lambda$, one has by definition of the Orlicz norm

$$\int_{[0, 1]^d} \psi_2(|B_n(x, \omega)|/2\sigma\lambda) \, dx > 1;$$

hence Čebyshev's inequality yields

$$\mathbb{P}(\omega : \|B_n(\cdot, \omega)\|_{\psi_2} > 2\sigma\lambda) \leqslant \frac{1}{\lambda^2}.$$

This implies

$$\mathbb{E}_\omega \|B_n(\cdot, \omega)\|_{\psi_2} = \int_0^\infty \mathbb{P}(\omega : \|B_n(\cdot, \omega)\|_{\psi_2} > t) \, dt \leqslant 2\sigma \left(1 + \int_1^\infty \frac{d\lambda}{\lambda^2}\right) = 4\sigma,$$

i.e., we have shown the assertion for $p = 2$. The case $2 < p < \infty$ can be proved by interpolation. Recall the well known estimate

$$\|f\|_{\psi_p} \leqslant \|f\|_{C([0,1]^d)}^{1-\theta} \|f\|_{\psi_2}^\theta$$

for all $f \in C([0, 1]^d)$, where

$$\frac{1}{p} = \frac{1-\theta}{\infty} + \frac{\theta}{2} = \frac{\theta}{2}.$$

Therefore, using Proposition 2 and Hölder's inequality we get

$$\begin{aligned}
\mathbb{E} \|B_n(\cdot)\|_{\psi_p} &\leqslant \mathbb{E} \|B_n(\cdot)\|_{C([0,1]^d)}^{1-\theta} \|B_n(\cdot)\|_{\psi_2}^\theta \\
&\leqslant (\mathbb{E} \|B_n(\cdot)\|_{C([0,1]^d)})^{1-\theta} (\mathbb{E} \|B_n(\cdot)\|_{\psi_2})^\theta \\
&\leqslant C(n^{d/2} 2^{-n/2})^{1-\theta} (n^{(d-1)/2} 2^{-n/2})^\theta \\
&\leqslant C n^{d/2 - 1/p} 2^{-n/2}
\end{aligned}$$

as asserted. ∎

Using the same technique as for the sup-norm we can derive the following results on Kolmogorov and entropy numbers and metric entropy of T_d with respect to the Orlicz norms.

THEOREM 5. *For all $2 \leqslant p < \infty$ and $d \geqslant 2$ there is a constant $C = C(p, d)$ such that for $T_d : L_2([0, 1]^d) \to L_{\psi_p}([0, 1]^d)$ and all $k \in \mathbb{N}$ the estimates*

$$d_k(T_d) \leqslant \frac{C}{k} (1 + \log k)^{d-1/2-1/p} \qquad and \qquad e_k(T_d) \leqslant \frac{C}{k} (1 + \log k)^{d-1/2-1/p}$$

hold. This implies the existence of a constant $C = C(p, d)$ such that

$$H(\varepsilon, T_d(B_{L_2([0,1]^d)})) \leqslant \frac{C}{\varepsilon} |\log \varepsilon|^{d-1/2-1/p}$$

for sufficiently small $\varepsilon > 0$.

3. LOWER BOUNDS OF THE SMALL BALL PROBABILITIES OF THE BROWNIAN SHEET

3.1. *Definitions and Results*

Consider the centered Gaussian process $\mathbb{B}_d := (B_x)_{x \in [0,1]^d}$ with covariance

$$\mathbb{E} B_x B_y = \prod_{j=1}^{d} \min(x_j, y_j)$$

where $x = (x_1, \ldots, x_d)$ and $y = (y_1, \ldots, y_d)$. This process is often called Brownian sheet. Other authors refer to it as multiparameter Brownian motion or Kolmogorov–Čentsov field. In applications the Brownian sheet tied down in the point $(1, \ldots, 1)$ plays an important role. In order to obtain this process one has to alter the covariance as follows $\mathbb{E} B_x' B_y' = \prod_{j=1}^{d} \min(x_j, y_j) - \prod_{j=1}^{d} x_j y_j$. Finally, one can also investigate a modification of the process which is zero in all points of the boundary of the unit cube, i.e., $\mathbb{E} B_x'' B_y'' = \prod_{j=1}^{d} (\min(x_j, y_j) - x_j y_j)$, which can be regarded as multiparameter generalization of the Brownian bridge. Our methods below give the same result for all three definitions since they differ just by "$d-1$"-parameter processes. Note that there exist a completely different multiparameter generalization of the Brownian motion, namely $(W_x)_{x \in [0,1]^d}$ with $\mathbb{E} W_x W_y = (|x|_2 + |y|_2 - |x - y|_2)/2$, which has a different small ball behavior (see, e.g., [24]).

It is known that the sample paths of $\mathbb{B}_d$ are a.s. continuous. We can consider them as random elements of the space $C([0,1]^d)$. It is a natural question to ask for the asymptotic behavior of the small ball probabilities

$$\mathbb{P}(\|\mathbb{B}_d\|_{C([0,1]^d)} < \varepsilon) = \mathbb{P}(\sup_{x \in [0,1]^d} |B_x| < \varepsilon) \tag{3}$$

as ε tends to zero. Such probabilities (of the tied down Brownian sheet) are important for power estimation for Kolmogorov–Smirnov and Kolmogorov statistics (see [1]). The same question arises if one wants to extend Chung's law of the iterated logarithm (cf. [28] for $d = 2$) to the Brownian sheet.

The asymptotic behavior of (3) is well known for $d = 1$, where we deal with the classical Wiener process. In this case, using techniques of differential equations one has several series representations for the probability (3), see, e.g., [12, Vol. 2, Chap. 10]. In particular,

$$\log \mathbb{P}(\|\mathbb{B}_1\|_{C([0,1])} < \varepsilon) \sim -\frac{\pi^2}{8\varepsilon^2}.$$

We refer to [4] for L_p-extensions of this result.

In the multiparameter case $d > 1$ it is not clear how to use differential equations and a completely different technique is needed. First estimates for (3) in the multiparameter case date back only to 1979 when Révész [22] proved for $d = 2$ the existence of constants C_1, C_2 such that

$$\exp\left(-\frac{C_1}{\varepsilon^2}|\log\varepsilon|^5\right) \leqslant \mathbb{P}(\|\mathbb{B}_2\|_{C([0,1]^2)} < \varepsilon) \leqslant \exp\left(-\frac{C_2}{\varepsilon^2}|\log\varepsilon|\right) \quad (4)$$

for small ε. In 1982, Csáki [8] found the asymptotic behavior of the small ball probability of $\mathbb{B}_d$ under the $L_2([0,1]^d)$-norm. He showed that

$$\log\mathbb{P}(\|\mathbb{B}_d\|_{L_2([0,1]^d)} < \varepsilon) \sim -\frac{K_d^2}{\varepsilon^2}|\log\varepsilon|^{2d-2}$$

with constant

$$K_d := \frac{2^{d-2}}{\sqrt{2}\,\pi^{d-1}(d-1)!}.$$

See [15] for various non-Brownian multiparameter generalizations of this result. Using the inequality $\|\cdot\|_{L_2([0,1]^d)} \leqslant \|\cdot\|_{C([0,1]^d)}$ one obtains

$$\mathbb{P}(\|\mathbb{B}_d\|_{C([0,1]^d)} < \varepsilon) \leqslant \exp\left(-\frac{C}{\varepsilon^2}|\log\varepsilon|^{2d-2}\right),$$

which improves in the case $d = 2$ the upper estimate in (4). The next result on this problem gave improved lower bounds. In 1986, Lifshits ([18], for $d = 2$) and in 1988, Bass ([1], for general d) obtained

$$\exp\left(-\frac{C}{\varepsilon^2}|\log\varepsilon|^{3d-3}\right) \leqslant \mathbb{P}(\|\mathbb{B}_d\|_{C([0,1]^d)} < \varepsilon). \quad (5)$$

At that stage, a considerable gap of order $d-1$ remained between the exponents of the log-terms in lower and upper bounds. In 1994, Talagrand [28] succeeded in proving the sharpness of (5) for $d = 2$. Later on other authors adapted the methods of Bass and Talagrand to other processes (see, e.g., [23, 30]). Surprisingly, the methods of [28] meet intrinsic difficulties when one tries to apply them for dimension $d > 2$.

Therefore, the question about sharp bounds for the small ball probabilities of $\mathbb{B}_d$ for $d \geqslant 3$ remains open. The conjecture in Remark 1.1 of [23] that (5) is also sharp for $d \geqslant 3$ is wrong. Indeed, using entropy technique from [13] we obtain the following lower bound.

THEOREM 6. *Let $d \geqslant 2$. For some constant C and all sufficiently small $\varepsilon > 0$, we have*

$$\exp\left(-\frac{C}{\varepsilon^2}|\log \varepsilon|^{2d-1}\right) \leqslant \mathbb{P}(\|\mathbb{B}_d\|_{C([0,1]^d)} < \varepsilon).$$

For $d = 2$ this bound just reproduces (5), while for $d \geqslant 3$ it improves (5) considerably and reduces the gap between the exponents of the log-terms in lower and upper bounds to one, independently of the dimension d:

$$\exp\left(-\frac{C_1}{\varepsilon^2}|\log \varepsilon|^{2d-1}\right) \leqslant \mathbb{P}\left(\|\mathbb{B}_d\|_{C([0,1]^d)} < \varepsilon\right) \leqslant \exp\left(-\frac{C_2}{\varepsilon^2}|\log \varepsilon|^{2d-2}\right),$$

We believe that one of these bounds must be sharp but we are still not sure which one (recall that for $d = 2$ the lower bound is sharp). For investigations considering other norms, as, e.g., Hölder norms, we refer the interested reader to [24].

Entropy Bounds for Small Ball Probabilities

We recall first the basic relations between the Brownian sheet, the operator T_d, and the results from [13], which we will use. Let $T_d^* : C^*([0,1]^d) \to L_2([0,1]^d)$ be the dual operator of T_d. One easily verifies that $\mathbb{E} B_x B_y = \langle T_d T_d^* \delta_x, \delta_y \rangle$ where δ_x denotes the Dirac measure in the point $x \in [0,1]^d$. It follows that the covariance operator of the Brownian sheet $\mathbb{B}_d$ equals $T_d T_d^*$. Hence, the reproducing kernel Hilbert space $H_{\mathbb{B}_d} \subset C([0,1]^d)$ possesses the representation $H_{\mathbb{B}_d} = T_d(L_2([0,1]^d))$ (see [17, Sect. 9, Theorem 4]). Denote by $K_{\mathbb{B}_d}$ its unit ball $T_d(B_{L_2([0,1]^d)})$ and define

$$\varphi(\varepsilon) := -\log \mathbb{P}(\|\mathbb{B}_d\|_{C([0,1]^d)} < \varepsilon).$$

It follows from general results of [13] (see [14, Theorem 7.6]) that for small $\varepsilon > 0$,

$$\varphi(\varepsilon) \leqslant C_1 \, H\left(\frac{C_2 \varepsilon}{\sqrt{\varphi(\varepsilon/2)}}, K_{\mathbb{B}_d}\right). \tag{6}$$

Moreover, under the additional assumption

$$\limsup_{\varepsilon \to 0} \frac{\varphi(\varepsilon)}{\varphi(2\varepsilon)} < \infty \tag{7}$$

our Corollary 3 and Theorem 2 from [13] would give the desired result of Theorem 6,

$$\varphi(\varepsilon) \leqslant \frac{C}{\varepsilon^2} |\log \varepsilon|^{2d-1}.$$

However, we do not know how to verify (7) and suggest the following iterative procedure based only on (6).

Proof of Theorem 6. Observe that (5) implies $\varphi(\varepsilon) \leqslant C_0 \varepsilon^{-3}$ as ε tends to zero. Then from Corollary 3 and (6) it follows that

$$\varphi(\varepsilon) \leqslant \sqrt{\varphi(\varepsilon/2)} \, \frac{C_1}{\varepsilon} \left(\log \frac{\sqrt{\varphi(\varepsilon/2)}}{C_2 \varepsilon} \right)^{d-1/2} \leqslant \sqrt{\varphi(\varepsilon/2)} \, \frac{C}{\varepsilon} \, |\log \varepsilon|^{d-1/2}.$$

Denoting $f(\varepsilon) := C \varepsilon^{-1} |\log \varepsilon|^{d-1/2}$ the inequality above reads as

$$\varphi(\varepsilon) \leqslant f(\varepsilon) \sqrt{\varphi(\varepsilon/2)}. \tag{8}$$

Since there is a constant $C_3 > 0$ such that $f(\varepsilon/2) \leqslant C_3 f(\varepsilon)$, iteration of (8) yields

$$\varphi(\varepsilon) \leqslant (\varphi(\varepsilon 2^{-N}))^{2^{-N}} C_3^{\sum_{n=1}^{N-1} 2^{-n}} f(\varepsilon)^{\sum_{n=0}^{N-1} 2^{-n}}$$

for all $N \in \mathbb{N}$. Since $\varphi(\varepsilon) \leqslant C_0 \varepsilon^{-3}$, the first factor tends to 1 as $N \to \infty$. Thus we get

$$\varphi(\varepsilon) \leqslant C_3 f(\varepsilon)^2 \leqslant \frac{C_4}{\varepsilon^2} |\log \varepsilon|^{2d-1}$$

for all sufficiently small $\varepsilon > 0$. ∎

Remark. We refer to a forthcoming paper of Li and Linde [16], where this iteration procedure will be further developed.

Orlicz case

Using the same technique and Theorem 5 we obtain the following lower bounds for the small ball probabilities of the Brownian sheet $\mathbb{B}_d$, $d \geqslant 2$, under the Orlicz norms considered in Section 2.

THEOREM 7. *Given* $2 \leqslant p < \infty$, *there exists a constant* $C = C(p, d)$ *such that we have*

$$\exp \left(-\frac{C}{\varepsilon^2} |\log \varepsilon|^{2d-1-2/p} \right) \leqslant \mathbb{P}(\|\mathbb{B}_d\|_{\psi_p} < \varepsilon)$$

for all sufficiently small $\varepsilon > 0$.

Remark. For $d = 2$, Talagrand [28] showed that there is an upper estimate of the same order. For $d \geqslant 3$ we only have the sharpness of these estimates for $p = 2$ (use [8] and observe that $\|\cdot\|_{\psi_2} \geqslant C \|\cdot\|_{L_2([0,1]^d)}$), for the remaining p's we do not know the precise behavior of $\mathbb{P}(\|\mathbb{B}_d\|_{\psi_p} < \varepsilon)$.

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